A Regular Pattern Among Primes

Michael J Mezzino Jr [1] **and Gary D Holtzclaw** [2]

[1] Emeritus Professor of Mathematics, University of Houston - Clear Lake
2700 Bay Area Blvd., Houston, TX 77058, United States
e-mail: mezzino@uhcl.edu

[2] Department of Decision Sciences (ret), University of Houston - Clear Lake
2700 Bay Area Blvd., Houston, TX 77058, United States
e-mail: holtzclawgary@gmail.com

Abstract: This paper presents a new prime-based cyclic pattern which suggests a possible relationship between Goldbach's Strong Conjecture, twin primes, and cousin primes.

Keywords: Pattern, twin and cousin primes, logical vector

1 Introduction

The following sets are used in this paper:

$$\mathcal{P} = \text{the set of odd prime numbers,}$$
$$\mathcal{E} = \text{the set of even integers reater than 4.}$$

The table below summarizes the results presented in this paper where the letter $e \in \mathcal{E}$. A relationship between e and twin primes is well known but a possible twin prime connection between e and $e+2$ appears new. The important cycles are related to the set $e \pmod 6 \equiv \{0, 2, 4\}$, written as $\acute{e}$ in col 3, where a new pattern, verified to 10^9, is presented in col 6 in which an intriguing relationship exists between Goldbach, twin primes, and cousin primes.

Patterns					
e	k	$é$	$\mathbf{R}_k$	Shift	$\triangle p$, $\triangle q$
6	1	0	↘•	$(3,3) \to (5,3)$	$2\,,0$
8	2	2	1↙•	$(5,3) \to (3,7)$	$-2\,,4$
10	3	4	↘•11	$(3,7) \to (5,7)$	$2\,,0$
12	4	0	0↘•10	$(5,7) \to (7,7)$	$2\,,0$
14	5	2	10↙•01	$(7,7) \to (5,11)$	$-2\,,4$
16	6	4	1•00↘•1	$(11,5) \to (13,5)$	$2\,,0$
18	7	0	0110↘••0	$(11,7) \to (13,7)$	$2\,,0$
20	8	2	10↙•00•01	$(7,13) \to (5,17)$	$-2\,,4$
22	9	4	1•00↘•0011	$(11,11) \to (13,11)$	$2\,,0$
24	10	0	0110↘••0110	$(11,3) \to (13,3)$	$2\,,0$
26	11	2	10↙•00•00101	$(7,19) \to (5,23)$	$-2\,,4$
28	12	4	0•00↘•0010010	$(11,17) \to (13,17)$	$2\,,0$
30	13	0	0010↘••0110100	$(11,19) \to (13,19)$	$2\,,0$
32	14	2	10000↙•00100001	$(13,19) \to (11,23)$	$-2\,,4$
34	15	4	1100↘•0010010011	$(11,23) \to (13,23)$	$2\,,0$
36	16	0	01100•0↘•10100110	$(17,19) \to (19,19)$	$2\,,0$
38	17	2	00100000↙•00000100	$(19,19) \to (17,23)$	$-2\,,4$
40	18	4	1000100↘•0010010001	$(17,23) \to (19,23)$	$2\,,0$
42	19	0	0100↘•100•0100110010	$(11,31) \to (13,31)$	$2\,,0$
44	20	2	10↙•00•00000000100101	$(7,37) \to (5,41)$	$-2\,,4$
46	21	4	1•00000↘•0010010000011	$(17,39) \to (19,39)$	$2\,,0$
48	22	0	0110↘•001•0000110010110	$(11,37) \to (13,37)$	$2\,,0$
50	23	2	10100•00100000100100101		

Observation: $\forall\ 6 \le e \le 10^9\ \exists\ p, q \in \mathcal{P} \ni e = p + q$ and

$e \,(\mathrm{mod}\, 6) \equiv 0 \to (p+2) \in \mathcal{P}$ and $e + 2 = (p+2) + q$,
$e \,(\mathrm{mod}\, 6) \equiv 2 \to (p-2)$ and $(q+4) \in \mathcal{P}$ and $e + 2 = (p-2) + (q+4)$,
$e \,(\mathrm{mod}\, 6) \equiv 4 \to (p+2) \in \mathcal{P}$ and $e + 2 = (p+2) + q$.

2 Patterns

When $e \in \mathcal{E}$ and $e = p+q$ for some $p, q \in \mathcal{P}$, we call the pair $\{p, q = e-p\}$ a *Goldbach Partition* (GP)[†] for e. A cursory scan of the columns suggests several potential and intriguing patterns developed from sequences of GPs. First consider the sequence (waterfall) of vectors in the fourth column representing GPs for $6 \leq e \leq 50$ where the $\bullet$ symbol indicates a value 1 that is integral to the waterfall pattern. Although other examples can also been observed, those rendered serve as sufficient examples to support the results obtained in this paper. The fifth and sixth columns also contain patterns which will be explained in the subsequent sections. For each $e \in \mathcal{E}, k = (e-6)/2+1$, each $\mathbf{R}_k$ will eventually be identified with a logical vector representing all GPs for e. Moreover, as we move from k to $k+1$, the sequence exhibits a partial left-shift or right-shift pattern indicated by the arrows and this relationship has been tested to hold for every $6 \leq e \leq 10^9$.[‡] This pattern suggests that if $e = p+q$ for $p, q \in \mathcal{P}$, then either $p+2$ or $p-2 \in \mathcal{P}$ and $p+2$ or $p-2 \in e+2$.

3 A Unique Logical Vector

To unpack this Table, we begin by defining a unique logical vector indexed by the odd integers greater than 1.

$$\mathbf{A} = (1_3, 1_5, 1_7, 0_9, 1_{11}, 1_{13}, 0_{15}, 1_{17}, \ldots, b_{2n+1}, \ldots), \tag{1}$$

where we define the binary coordinate $b_{2n+1},\ 1 \leq n$, to be 1 when $2n+1$ is a prime and 0 otherwise. For simplicity, we will drop the subscripts of odd integers greater than 1 on all $\mathbf{A},\ 1 \leq i$ vectors and consider them implied from now on. We define an injection mapping $\mathcal{I} : \{(b_1, b_2, b_3, \ldots, b_k \ldots) : b_i \in \{0,1\}\} \to \mathcal{P}$ by

$$\mathcal{I}(b_i) = p_i \textit{ only when } b_i = 1. \tag{2}$$

$\mathcal{I}(\mathbf{A}) = \mathcal{P}$ follows immediately from (1) and the definition of the injection mapping $\mathcal{I}$. For example, $\mathbf{R}_5 = (1,0,1,0,1)$ implies $\mathcal{I}(\mathbf{R}_5) = \{3, 7, 11\}$.

[†] https://mathworld.wolfram.com/GoldbachPartition.html

[‡] A C++ program capable of performing calculations with several output options relevant to this paper, including the procedure in the Table, is available from github by evaluating "git clone https://github.com/mezzino/Goldbach-Patterns". The README file includes instructions for executing the routine and interpreting the results. We used this C++ program to verify these ideas for $6 \leq e \leq 10^9$.

4 Logical Paring

From this point on we will assume that $e \in \mathcal{E}$ and concentrate on the odd primes and their double-primes, starting with $e = 6 = 2 \cdot 3$. If 3 is one of the primes in its GP, then the other prime must be $e - 3$, or equivalently, in terms of double primes, $e = (2 \cdot 3 + 2 \cdot (e-3))/2$ and $k = (e-6)/2 + 1$. This latter expression shows that e can also be viewed as the mid-point of a line segment between two double primes for every GP. Hence we introduce the following notation

$$\mathbf{A}_k = \mathbf{A} \cap \{(b_1, \ldots, b_k) : b_i \in \{0,1\}\} = \{(1,1,1,0,\ldots,b_k) : b_k \in \{0,1\}\}. \quad (3)$$

Observe that the injection mapping $\mathcal{I}$ identifies $\mathbf{A}_k$ with the complete sequence of primes $\mathcal{P}$. Correspondingly, we define

$$\mathbf{B}_k = \{(b_k, \ldots, 0, 1, 1, 1) : b_k \in \{0,1\}\}, \quad (4)$$

obtained from $\mathbf{A}_k$ by reversing the coordinates. It will be convenient to state that for any vector $\vec{v}$, we define $(\vec{v})_i$ to be the i^{th} coordinate of $\vec{v}$, and define the coordinate-wise conjunction of $\mathbf{A}_k$ and $\mathbf{B}_k$ by

$$\mathbf{A}_k \otimes \mathbf{B}_k = (\mathbf{R}_k)_i = \begin{cases} 1 & \text{if } (\mathbf{A}_k)_i \,\&\, (\mathbf{B}_k)_i = (\mathbf{A}_k)_i \cdot (\mathbf{B}_k)_i = 1, \ 1 \le i \le k, \\ 0 & \textit{otherwise}. \end{cases} \quad (5)$$

If the interest is only in the existence of at least one solution, then, $\mathbf{A}_k \otimes \mathbf{B}_k \neq \mathbf{O}$ is equivalent to the inner product $\langle \mathbf{A}_k, \mathbf{B}_k \rangle \neq 0$. Therefore, a solution will fail to exist if and only if $\mathbf{A}_k \perp \mathbf{B}_k$, or equivalently, $\mathbf{B}_k \in \mathbf{A}_k^{\perp}$.

Beginning with $e = 6, k = 1$, we obtain $\mathbf{A}_1 = \mathbf{B}_1 = \mathbf{R}_1 = (1)$ and for $e = 8, k = 2$, $\mathbf{A}_2 = \mathbf{B}_2 = \mathbf{R}_2 = (11)$. Continuing, let $e = 30, k = 13$ and

$$\begin{aligned} \mathbf{A}_{13} &= (1,1,1,0,1,1,0,1,1,0,1,0,0), \\ \mathbf{B}_{13} &= (0,0,1,0,1,1,0,1,1,0,1,1,1). \end{aligned} \quad (6)$$

The complete space of partitions for $e = 30$ is obtained from

$$\mathbf{A}_{13} \otimes \mathbf{B}_{13} = \mathbf{R}_{13} = (0,0,1,0,1,1,0,1,1,0,1,0,0), \quad (7)$$

and $\mathcal{I}(\mathbf{R}_{13}) = \{7, 11, 13, 17, 19, 23\}$ is precisely the complete set of values which when paired represent all the GPs for $e = 30$. The corresponding pairing applied to $\mathcal{I}(\mathbf{R}_{13})$ yields

$$\underbrace{7 + \underbrace{11 + \underbrace{13 + 17}_{30} + 19}_{30} + 23}_{30} . \tag{8}$$

That is, not only is *a* partition found but, at least in this case, *every* partition is found. Also, e will be a double-prime whenever $|\mathcal{I}(\mathbf{R}_k)|$ is odd as seen for $e = 14,\ k = 5$ with $\mathcal{I}(\mathbf{R}_5) = (3, 0, 7, 0, 11)$, yielding $e = 7+7 = 3+11$.

For any $k > 0$, the result $\mathbf{A}_k \otimes \mathbf{B}_k \neq \mathbf{O}$ suggests that $\mathbf{A}_k$ contains a subset, identified by $\mathbf{R}_k$, which is a mirror image of the corresponding subset in $\mathbf{B}_k$. Our core approach includes the principle of *mirror symmetry*. This example suggests the following conjecture:

Conjecture: $\forall e \in \mathcal{E},\ k = (e - 6)/2 + 1 \to \mathbf{A}_k \otimes \mathbf{B}_k = \mathbf{R}_k \neq \mathbf{O}$.

If $|\mathbf{R}_k|$ is odd, examine the point at the center. If this point is not zero, then e is a double prime and we are done. Otherwise, begin with the centered pair. If both are not zero, then e has a GP, as indicated in the preceding example. If this condition fails, continue moving out maintaining symmetry from the starting point. Clearly, Goldbach's Strong Conjecture is equivalent to this procedure always succeeds.

5 Twin Primes

Consider the natural symmetry of

$$\mathbf{A}_k \otimes \mathbf{B}_k = \mathbf{R}_k = (b_1, b_2, \ldots, b_i \ldots, b_{k-i+1}, \ldots, b_{k-1}, b_k), \tag{9}$$

with $b_j = b_{k-j+1}, 1 \leq j \leq k$. A pattern within the sequence of $\mathbf{R}_k$s is seen in the Table. Whenever $b_i = 1,\ 1 \leq i \leq k$, symmetry implies the following pairing:

$$\underbrace{(\underbrace{b_1, b_2, \ldots, \underbrace{b_i, \ldots, b_{k-i+1}} \ldots, b_{k-1}}, b_k)} . \tag{10}$$

In addition, GPs can be determined from

$$\mathcal{I}(\mathbf{R}_k) = \{p_1, p_2, \ldots, p_i \ldots, p_{\hat{k}-i+1}, \ldots, p_{\hat{k}-1}, p_{\hat{k}}\},\ \hat{k} < k. \tag{11}$$

where, in general, $\hat{k}$ primes are obtained. We now take s determined by the symmetry indicated in (12) as follows:

$$\{p_1, p_2, \ldots, \underbrace{p_j, \ldots, p_{\hat{k}-j+1}}_{d_j}, \ldots, p_{\hat{k}-1}, p_{\hat{k}}\} \tag{12}$$

$$\underbrace{\qquad p_2, \ldots, p_{\hat{k}-1} \qquad}_{d_2} \qquad \underbrace{\qquad p_1, \ldots, p_{\hat{k}} \qquad}_{d_1}$$

where $1 \leq j \leq \lfloor(\hat{k}+1)/2\rfloor$ represents the index of the smallest prime in each difference. In general, we obtain $\{d_1, d_2, \ldots, d_j, \ldots, d_{\lfloor(\hat{k}+1)/2\rfloor}\}$ where each d_j is associated with a $p_j \in \mathbf{R}_k$. In addition to GPs conjectured by the previous analysis, a closer examination of the Table exposes a deeper relationship between these differences at e and at $e+2$.

Curiously, for most values of e there exists a non-zero value at the i^{th} coordinate for which a non-zero coordinate exists at either $i+1$ or $i-1$ at $e+2$. This is equivalent to saying that for each e, there exists a $d_i(e) \neq 0$ where either $d_{i-1}(e+2)$ or $d_{i+1}(e+2)$ are non-zero. Assuming $p \leq q$, since, $d_i(e) = q - p$, then either $d_{i+1}(e+2) = d_i(e) - 2 = q - (p+2)$ or $d_{i-1}(e+2) = d_i(e) + 6 = (q+4) - (p-2)$. From the prior analysis of the conjectured structure of $\mathbf{R}_k$, then either $p+2 \in \mathcal{P}$ or $p-2$ and $q+4 \in \mathcal{P}$ reflected in the Table's sixth column. Also, for the procession from e to $e+2$, d_i either increases by 6 or decreases by 2 for the appropriate case. Hence, in either case, $(e+2)$ admits a GP. Also, there is a predictability to this phenomenon.

Continuing with the example for $e = 30,\ k = 13$ together with $e = 32,\ k = 14$, we have $30 \pmod 6 \equiv 0$, and

$$\begin{aligned} \mathbf{R}_{13} &= 0010 \bullet\!\!\searrow 10110100 \ \rightarrow\ 30 = 11 + 19 \ (d_5 = 8), \\ \mathbf{R}_{14} &= 10000 \bullet 00100001 \rightarrow 32 = 13 + 19 \ (d_6 = 6), \end{aligned} \tag{13}$$

where the indexes correspond to the positions of the relevant primes. In addition, the transition of the differences from d_5 to d_6 yielding $d_5 - d_6 = 2 = \triangle q - \triangle p$ is directly related to column 6 in the Table. In fact, it is these differences which initially led to discovering the patterns reflected in the Table. For $e = 32,\ k = 14$ together with $e = 34,\ k = 15$, we have $34 \pmod 6 \equiv 2$, and

$$\begin{aligned} \mathbf{R}_{14} &= 10000\swarrow 00100001 \ \rightarrow 32 = 13 + 19 \ (d_6 = 6), \\ \mathbf{R}_{15} &= 1100 \bullet 0010010011 \rightarrow 34 = 11 + 23 \ (d_5 = 12). \end{aligned} \tag{14}$$

Note that for $e = 34,\ k = 15$ together with $e = 36,\ k = 16$, we have $34 \pmod 6 \equiv 4$ but we do not select the obvious transition $5 \rightarrow 7$. Including sequential solutions beginning with $p = 5$ occasionally breaks the $e \pmod 6$ cycle of $0, 2, 4$.

$$\begin{aligned} \mathbf{R}_{15} &= 1100\searrow 0010010011 \rightarrow \ 34 = 11 + 23 \ (d_5 = 12), \\ \mathbf{R}_{16} &= 01100 \bullet 0110100110 \rightarrow 36 = 13 + 23 \ (d_6 = 10). \end{aligned} \tag{15}$$

For $6 \leq e \leq 10^9$ with $e = p + q,\ p \neq 5$, we have observed this phenomenon remaining consistent as $e \pmod 6$ cycles through $0,\ 2,\ 4$. Note that if you allow $p = 5$, the only prime which both is a twin and has a twin, the cycle is frequently not preserved. Also, when $e \pmod 6 \equiv 0$ or 4 except when $p = 3,\ p$ cannot end in 3 because $p + 2$ would end in 5 and not be a prime and when $e \pmod 6 \equiv 2$ except when $p = 7,\ p$ cannot end in 7 and q cannot end in 1 again because $p - 2$ and $q + 4$ would end in 5 and not be prime.

6 Conclusion

The existence of infinitely many twin prime numbers allows us to ensure that there exists infinitely many even numbers $e = 2n$ such that $e - 1$ and $e + 1$ are twin prime numbers. Hence, this implies that infinitely many even numbers can be written as the sum of two twin prime numbers, and we will have proved one important piece of Goldbach's Strong Conjecture. In addition, our observation implies a surprisingly regular pattern $\bmod 6$. That is, for each $e \pmod 6$ option, at least one solution from e to $e + 2$ is obtained using the technique suggested in the observation. In addition, for every $e > 6$ there exists at least one p such that $e/10 < p$ which satisfies both the twin prime and cousin prime conjectures. Interestingly, since $p + 2$, or $p - 2$, or $q + 4$ appear in this discussion, the theory of twin primes and cousin primes may share another curious relationship to Goldbach's Strong conjecture.

Declaration of Interest

No potential conflicts of interest were reported by the authors.

Funding

No external funding was available to the authors.

References

[1] Apostol, T. (1976) *Introduction to Analytic Number Theory.* Springer Link, New York, NY.
[2] Ball, W. W. R., Coxeter, H. S. M. *Mathematical Recreations and Essays.* Dover Publications, New York, NY.
[3] Guy, R. K. (2010) *Unsolved Problems in Number Theory - Third Edition.* Springer, New York, NY.
[4] Nambiar, K.K. (2002). *Geometrical Equivalents of Goldbach Conjecture and Fermat Like Theorem.* Science Direct Working Paper No S1574-0358(04)70371-8,
Available at SSRN: https://ssrn.com/abstract=3142729
[5] Oliveira e Silva, T., Herzog, S., and Pard, S., (2014). *Empirical verification of the even Goldbach conjecture and computation of prime gap up to* 4×10^{18}. Mathematrics of Computation, 83 , pp. 2033-2060.
[6] Orús–Lacort M, Orús R, Jouis C (2024) *Analyzing twin primes, goldbach's strong conjecture and polignac's conjecture.* Ann Math Phys 7(1): 076-084. DOI: 10.17352/amp.000110.
[7] Pintz, J. (2007) *Approximations to the Goldbach and twin prime problem and gaps between consecutive primes.* Advanced Studies in Pure Mathematics, 49, 2007, Probability and Number Theory-Kanazawa 2005, pp. 323-365.
[8] Rosen, K. H. (2011) *Elementary Number Theory - Sixth Edition.* Addison-Wesley, Boston, MA.
[9] Sheikh, N. Z. (2021) *A Pair of Formulae to the Goldbach's Strong Conjecture.* Journal of Applied & Computational Mathematics. 10 (8).
[10] Tao, T. (2014) *Bounded gaps between primes.* Polymath.
[11] Tao, T. (2011) *Structure and Randomness in the Prime Numbers.* YouTube, [Online]. Available:
https: //www.youtube.com/watch?v=hYxBH1YY9z4&t=1976s.
[12] Wang, Y. (Ed.) (2002) *The Goldbach Conjecture - Second Edition.* World Scientific Publishing Co., Hackensack, New Jersey.

[13] Wolfram, S. (2022). *Mathematica* Version 14. Wolfram Media, Inc., Champaign, IL.

December 9, 2025